

# New Insights into the Perturbative Structure of Electroweak Sudakov Logarithms

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## Abstract

To match the expected experimental precision at future linear colliders, improved theoretical predictions beyond next-to-leading order are required. At the anticipated energy scale of  $\sqrt{s} = 1$  TeV the electroweak virtual corrections are strongly enhanced by collinear-soft Sudakov logarithms of the form  $\log^2(s/M^2)$ , with  $M$  being the generic mass scale of the  $W$  and  $Z$  bosons. By choosing an appropriate gauge, we have developed a formalism to calculate such corrections for arbitrary electroweak processes. As an example we consider in this letter the process  $e^+e^- \rightarrow f\bar{f}$  and study the perturbative structure of the electroweak Sudakov logarithms by means of an explicit two-loop calculation. In this way we investigate how the Standard Model, with its mass gap between the photon and  $Z$  boson in the neutral sector, compares to unbroken theories like QED and QCD. In contrast to what is known for unbroken theories we find that the Sudakov logarithms are not exclusively given by the so-called rainbow diagrams, owing to the mass gap and the charged-current interactions. In spite of this, we nevertheless observe that the two-loop corrections are consistent with an exponentiation of the one-loop corrections. In this sense the Standard Model behaves like an unbroken theory at high energies.

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# 1 Introduction

At the next generation of colliders center-of-mass energies will be reached that largely exceed the electroweak scale. For instance, the energy at a future linear  $e^+e^-$  collider is expected to be in the TeV range [1]. At these energies one enters the realm of large perturbative corrections. Even the effects arising from weak corrections are expected to be of the order of 10% or more [2, 3], i.e. just as large as the well-known electromagnetic corrections. In order not to jeopardize any of the high-precision studies at these high-energy colliders, it is therefore indispensable to improve the theoretical understanding of the radiative corrections in the weak sector of the Standard Model (SM). In particular this will involve a careful analysis of effects beyond first order in the perturbative expansion in the (electromagnetic) coupling  $\alpha = e^2/(4\pi)$ .

The dominant source of radiative corrections at TeV-scale energies is given by logarithmically enhanced effects of the form  $\alpha^n \log^m(M^2/s)$  for  $m \leq 2n$ , involving particle masses  $M$  well below the collider energy  $\sqrt{s}$ . A natural way of controlling the theoretical uncertainties would therefore consist in a comprehensive study of these large logarithms, taking into account all possible sources (i.e. ultraviolet, soft, and collinear). In first approximation the so-called Sudakov logarithms  $\propto \alpha^n \log^{2n}(M^2/s)$ , arising from collinear-soft singularities [4], constitute the leading contribution to the large electroweak correction factors. Recent studies have focused on these Sudakov effects in the process  $e^+e^- \rightarrow f\bar{f}$  [5]–[7]. Unfortunately the three independent studies are in mutual disagreement, exhibiting strikingly different higher-order results already for the virtual corrections. The main cause for the differences can be traced back to the use of different assumptions concerning the exponentiation properties of the Sudakov logarithms in the SM. Many of these assumptions are based on the analogy with unbroken theories like QED and QCD, where the resummation of the higher-order effects amounts to an exponentiation of the one-loop corrections (see for instance Refs. [4] and [8]–[10]). However, the SM is a broken gauge theory with a large mass gap in the neutral sector between the massless photon and the massive  $Z$  boson, making it a theory with more than one mass scale. As such it remains to be seen how much of the analogy with unbroken theories actually pertains to the SM.

In this letter we therefore focus on the virtual Sudakov logarithms in the reaction  $e^+e^- \rightarrow f\bar{f}$ , in an attempt to clarify how the  $\mathcal{O}(\alpha^2)$  effects relate to the  $\mathcal{O}(\alpha)$  ones. In this way we identify to what extent the SM behaves like an unbroken theory at high energies. Moreover, since the Sudakov logarithms originate from the exchange of soft, effectively on-shell gauge bosons, many of the features derived for the virtual corrections are intimately related to properties of the corresponding real-gauge-boson emission processes.

## 2 Electroweak Sudakov logarithms in $e^+e^- \rightarrow f\bar{f}$

In order to facilitate the calculation of the one- and two-loop Sudakov logarithms, we work in the Coulomb gauge for both massless and massive gauge bosons (the subtleties associated with the Coulomb gauge for massive gauge bosons will be discussed in Ref. [11]). Working in this special gauge has the advantage that all virtual Sudakov logarithms are contained exclusively in the self-energies of the external on-shell particles [9, 11] or the self-energies of any intermediate particle

that happens to be effectively on-shell.<sup>1</sup> The latter is, for instance, needed for the production of near-resonance unstable particles. The elegance of this method lies in its universal nature, with the Sudakov logarithms originating from vertex, box etc. corrections being suppressed. Once all self-energies to all on-shell/on-resonance SM particles have been calculated, the prediction of the Sudakov form factor for an *arbitrary* electroweak process becomes trivial. The relevant self-energies for the calculation of the Sudakov logarithms involve the exchange of collinear-soft gauge bosons, including their potential mixing with the corresponding would-be Goldstone bosons. The collinear-soft exchange of fermions and ghosts leads to suppressed contributions, since the propagators of these particles do not have the required pole structure.

## 2.1 The fermionic self-energy at one-loop level

As mentioned above, in order to determine the Sudakov logarithms in the process  $e^+e^- \rightarrow f\bar{f}$  one has to calculate the external self-energies (i.e. the wave-function factors) of all fermions involved in the process.

Consider to this end the fermionic one-loop self-energy  $\Sigma^{(1)}(p, n, M_1)$ , originating from the emission of a gauge boson  $V_1$  with loop-momentum  $k_1$  and mass  $M_1$  from a fermion  $f$  with momentum  $p$ :

$$-i \Sigma^{(1)}(p, n, M_1) = \text{Diagram showing a fermion line with momentum } p \text{ entering from the left, a loop with momentum } k_1 \text{ and mass } M_1 \text{ (labeled } V_1(k_1) \text{)}, \text{ and the fermion line exiting with momentum } p \text{ on the right. The internal fermion line is labeled } f_1(p-k_1) \text{ and the external lines are labeled } f(p) \text{ and } f(p). \text{ The diagram is represented as a horizontal line with two vertices connected by a wavy line (gauge boson) and a straight line (fermion). The left vertex is labeled } f(p) \text{ and the right vertex is labeled } f(p). \text{ The wavy line is labeled } V_1(k_1) \text{ and the straight line is labeled } f_1(p-k_1) \text{."}$$

Here  $n$  is the unit vector in the time direction, which enters by virtue of using the Coulomb gauge. In the high-energy limit the fermion mass in the numerator of the fermion propagator can be neglected and similarly the contribution involving a mixed gauge-boson – Goldstone-boson propagator can be discarded. The self-energy  $\Sigma^{(1)}$  then contains an odd number of  $\gamma$ -matrices, leading to the following natural decomposition in terms of the two possible structures  $\not{p}$  and  $\not{n}$ :

$$\Sigma^{(1)}(p, n, M_1) \approx \left[ \not{p} \Sigma_p^{(1)}(n \cdot p, p^2, M_1) + \not{n} \frac{p^2}{n \cdot p} \Sigma_n^{(1)}(n \cdot p, p^2, M_1) \right] e^2 \Gamma_{ffV_1}^2. \quad (1)$$

The coupling factor  $\Gamma_{ffV_1}$  is defined according to

$$\Gamma_{ffV_1} = V_{ffV_1} - \gamma_5 A_{ffV_1}, \quad (2)$$

where  $V_{ffV_1}$  and  $A_{ffV_1}$  are the vector and axial-vector couplings of the fermion  $f$  to the exchanged gauge boson  $V_1$ . In our convention these coupling factors read

$$\Gamma_{ff\gamma} = -Q_f, \quad \Gamma_{ffZ} = \frac{(1 - \gamma_5) I_f^3 - 2 Q_f \sin^2 \theta_w}{2 \cos \theta_w \sin \theta_w}, \quad \Gamma_{ffW} = \frac{(1 - \gamma_5)}{2\sqrt{2} \sin \theta_w}. \quad (3)$$

Here  $I_f^3$  is the third component of the weak isospin,  $e Q_f$  is the electromagnetic charge, and  $\theta_w$  is the weak mixing angle.

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<sup>1</sup>Note that similar simplifications can probably be obtained equally well by working in an axial gauge, see for instance Ref. [10] for massless particles.

The contribution to the external wave-function factor now amounts to multiplying the self-energy by  $i/\not{p}$  on the side where it is attached to the rest of the scattering diagram and by the appropriate fermion source on the other side. Finally the square root should be taken of the external wave-function factor, i.e. the one-loop contribution should be multiplied by the usual factor 1/2. For an initial-state fermion, for example, one obtains<sup>2</sup>

$$\begin{aligned} \frac{1}{2} \frac{i}{\not{p}} \left[ -i \Sigma^{(1)}(p, n, M_1) \right] u_f(p) &\approx \frac{e^2}{2} \Gamma_{ffV_1}^2 \left[ \Sigma_p^{(1)}(n \cdot p, m_f^2, M_1) + 2 \Sigma_n^{(1)}(n \cdot p, m_f^2, M_1) \right] u_f(p) \\ &\equiv \frac{1}{2} \Sigma_f^{(1)}(s, m_f^2, M_1) u_f(p) , \end{aligned} \quad (4)$$

where  $m_f$  is the mass of the external fermion and  $\sqrt{s} = 2p_0$  is the center-of-mass energy of the process  $e^+e^- \rightarrow f\bar{f}$ . Therefore the quantity of interest is actually  $\Sigma_f^{(1)}$ , which can be extracted from the full fermionic self-energy by means of the projection

$$\begin{aligned} \Sigma_f^{(1)}(s, m_f^2, M_1) &= \frac{i}{2p_0} \bar{u}_f(p) \left\{ \frac{\partial}{\partial p_0} \left[ -i \Sigma^{(1)}(p, n, M_1) \right] \right\} u_f(p) \\ &\approx -e^2 \Gamma_{ffV_1}^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{4 p_\mu p_\nu}{[(p - k_1)^2 - m_{f_1}^2 + i\epsilon]^2} P^{\mu\nu}(k_1, n) , \end{aligned} \quad (5)$$

containing the gauge-boson propagator in the Coulomb gauge

$$P^{\mu\nu}(k_1, n) = \frac{-i}{k_1^2 - M_1^2 + i\epsilon} \left[ g^{\mu\nu} + \frac{k_1^\mu k_1^\nu - k_1^0 (k_1^\mu n^\nu + n^\mu k_1^\nu)}{\vec{k}_1^2} \right] . \quad (6)$$

Note that the loop-momentum  $k_1$  has been neglected in the numerator of the fermion propagator, since only collinear-soft gauge-boson momenta will give rise to the Sudakov logarithms. The mass of the fermion inside the loop,  $m_{f_1}$ , is at best of the order of the  $Z$ -boson mass (for the top-quark). At the leading-logarithmic level it therefore only enters as an independent mass scale if the exchanged gauge boson is a photon (i.e.  $m_{f_1} = m_f$ ), where it is needed for the regularization of collinear singularities. Therefore we restrict the arguments of  $\Sigma_f^{(1)}$  to the energy and mass of the emitting fermion and the mass of the exchanged gauge boson, which are the three relevant scales for describing the Sudakov logarithms.

Having two canonical momenta at our disposal, i.e.  $p$  and  $n$ , we define the following Sudakov parametrisation of the gauge-boson loop-momentum  $k_1$ :

$$k_1 = v_1 q + u_1 \bar{q} + k_{1\perp} , \quad (7)$$

with

$$\begin{aligned} p^\mu &\equiv (E, \beta_f E, 0, 0) , & \beta_f &= \sqrt{1 - m_f^2/E^2} , & s &= 4 E^2 , \\ q^\mu &= (E, E, 0, 0) , & \bar{q}^\mu &= (E, -E, 0, 0) , & k_{1\perp}^\mu &= (0, 0, \vec{k}_{1\perp}) . \end{aligned} \quad (8)$$

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<sup>2</sup>For an outgoing fermion one obtains  $\frac{1}{2} \bar{u}_f(p) \bar{\Sigma}_f^{(1)}(s, m_f^2, M_1)$ , where  $\bar{\Sigma}_f^{(1)}$  can be derived from  $\Sigma_f^{(1)}$  by reversing the sign in front of  $\gamma_5$ .

In terms of this parametrisation, the integration measure  $d^4k_1$ , the invariants  $(p \cdot k_1)$  and  $k_1^2$ , and the gauge-boson energy  $k_1^0$  read

$$\begin{aligned} d^4k_1 &= \pi \frac{s}{2} dv_1 du_1 d\vec{k}_{1\perp}^2, \\ (p \cdot k_1) &= \frac{s}{4} [v_1 (1 - \beta_f) + u_1 (1 + \beta_f)] \approx \frac{s}{2} \left( u_1 + \frac{m_f^2}{s} v_1 \right), \\ k_1^2 &= s v_1 u_1 - \vec{k}_{1\perp}^2 \quad \text{and} \quad k_1^0 = \frac{\sqrt{s}}{2} (v_1 + u_1). \end{aligned} \quad (9)$$

The term containing the fermion mass  $m_f$  is needed for the exchange of photons only, regulating the collinear singularity at  $u_1 = 0$ . For the exchange of a massive gauge boson the mass  $M_1$  will be the dominant collinear as well as infrared regulator.

The  $v_1$ -integration is restricted to the interval  $0 \leq v_1 \leq 1$ , as a result of the requirement of having poles in both hemispheres of the complex  $u_1$ -plane. The residue is then taken in the lower hemisphere in the pole of the gauge-boson propagator:  $s v_1 u_1^{\text{res}} = \vec{k}_{1\perp}^2 + M_1^2 \equiv s v_1 y_1$ . Finally,  $\vec{k}_{1\perp}^2$  is substituted by  $y_1$ , with the condition  $\vec{k}_{1\perp}^2 \geq 0$  translating into  $v_1 y_1 \geq M_1^2/s$ . The one-loop Sudakov contribution to  $\Sigma_f^{(1)}$  now reads

$$\begin{aligned} \Sigma_f^{(1)}(s, m_f^2, M_1) &\approx -\frac{\alpha}{\pi} \Gamma_{ff_1 V_1}^2 \int_0^\infty dy_1 \int_0^1 dv_1 \frac{\Theta(v_1 y_1 - \frac{M_1^2}{s})}{(y_1 + \frac{m_f^2}{s} v_1) (v_1 + y_1)} \\ &\approx -\frac{\alpha}{\pi} \Gamma_{ff_1 V_1}^2 \int_0^1 \frac{dy_1}{y_1} \int_{y_1}^1 \frac{dz_1}{z_1} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1), \end{aligned} \quad (10)$$

with the integration kernel  $\mathcal{K}^{(1)}$  given by

$$\mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) = \Theta\left(y_1 z_1 - \frac{M_1^2}{s}\right) \Theta\left(y_1 - \frac{m_f^2}{s} z_1\right). \quad (11)$$

Here we introduced the energy variable  $z_1 = v_1 + y_1$  and made use of the fact that only collinear-soft gauge-boson momenta are responsible for the quadratic large-logarithmic effects:  $y_1, z_1 \ll 1$ . As a result, the gauge boson inside the loop is effectively on-shell and transversely polarized.

The exchanged gauge boson can either be a massless photon ( $\gamma$ ) or one of the massive weak bosons ( $W$  or  $Z$ ). The associated mass gap gives rise to distinctive differences in the two types of contributions (see Sect. 2.3). Bearing in mind that the SM is not parity conserving, we sum over the gauge-boson contributions and present the full one-loop Sudakov correction factor for right- and left-handed fermions/antifermions separately:

$$\delta_{f_R}^{(1)} = \delta_{\bar{f}_L}^{(1)} = \left( \frac{Y_f^R}{2 \cos \theta_w} \right)^2 L(M, M) + Q_f^2 \left[ L_\gamma(\lambda, m_f) - L(M, M) \right], \quad (12a)$$

$$\delta_{f_L}^{(1)} = \delta_{\bar{f}_R}^{(1)} = \left[ \frac{C_F}{\sin^2 \theta_w} + \left( \frac{Y_f^L}{2 \cos \theta_w} \right)^2 \right] L(M, M) + Q_f^2 \left[ L_\gamma(\lambda, m_f) - L(M, M) \right], \quad (12b)$$

with

$$L(M_1, M_2) = -\frac{\alpha}{4\pi} \log\left(\frac{M_1^2}{s}\right) \log\left(\frac{M_2^2}{s}\right), \quad (13)$$

$$L_\gamma(\lambda, M_1) = -\frac{\alpha}{4\pi} \left[ \log^2\left(\frac{\lambda^2}{s}\right) - \log^2\left(\frac{\lambda^2}{M_1^2}\right) \right]. \quad (14)$$

Note that these correction factors are the same for incoming as well as outgoing particles. In Eq. (12)  $Y_f^{R,L}$  denotes the right- and left-handed hypercharge of the external fermion, which is connected to the third component of the weak isospin  $I_f^3$  and the electromagnetic charge  $eQ_f$  through the Gell-Mann – Nishijima relation  $Q_f = I_f^3 + Y_f^{R,L}/2$ . The coefficient  $C_F = 3/4$  is the Casimir operator in the fundamental representation of  $SU(2)$  and  $\lambda$  is the fictitious (infinitesimally small) mass of the photon needed for regularizing the infrared singularity at  $z_1 = 0$ . For the sake of calculating the leading Sudakov logarithms, the masses of the  $W$  and  $Z$  bosons can be represented by one generic mass scale  $M$ . Note that the terms proportional to  $Q_f^2$  in Eq. (12) are the result of the mass gap between the photon and the weak bosons.

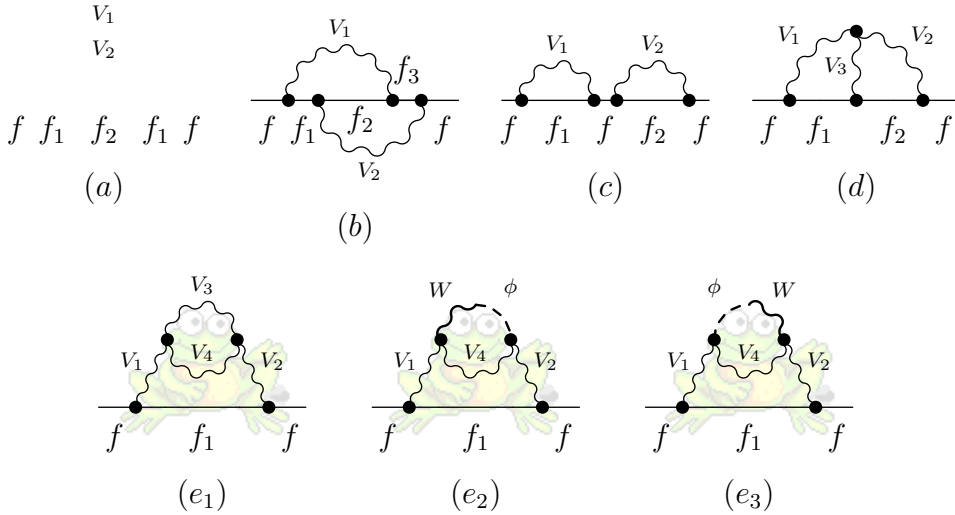
In the process  $e^+e^- \rightarrow f\bar{f}$  the one-loop correction factors presented in Eq. (12) contribute in the following way to the polarized matrix element, bearing in mind that at high energies the helicity eigenstates are equivalent to the chiral eigenstates:

$$\mathcal{M}_{e_R^+e_L^- \rightarrow f_L\bar{f}_R}^{1\text{-loop, sudakov}} = \frac{1}{2} \left[ \delta_{e_R^+}^{(1)} + \delta_{e_L^-}^{(1)} + \delta_{f_L}^{(1)} + \delta_{\bar{f}_R}^{(1)} \right] \mathcal{M}_{e_R^+e_L^- \rightarrow f_L\bar{f}_R}^{\text{born}}, \quad (15)$$

and similar expressions for the other possible helicity combinations.

## 2.2 The fermionic self-energy at two-loop level

At two-loop accuracy one has to take the following five generic sets of diagrams into account:



The fermions  $f_i$  are fixed by the exchanged gauge bosons  $V_i$ . Various cancellations are going to take place between all these diagrams. In unbroken theories like QED and QCD merely the so-called ‘rainbow’ diagrams of set (a) survive. The same holds if all gauge bosons of the theory would have a similar mass. The unique feature of the SM is that it is only partially broken,

with the electromagnetic gauge group  $U(1)_{\text{em}} \neq U(1)_Y$  remaining unbroken. As such three of the four gauge bosons will acquire a mass, whereas the photon remains massless and will interact with the charged massive gauge bosons ( $W^\pm$ ). As a consequence, merely calculating the ‘rainbow’ diagrams will *not* lead to the correct result, in contrast to what is claimed in Refs. [6, 7].

The generic two-loop contribution of Sudakov logarithms to  $\Sigma_f^{(2)}$  reads:

$$\Sigma_f^{(2)} \approx \left(-\frac{\alpha}{\pi}\right)^2 \Gamma_f^{(2)} \int_0^1 \frac{dy_1}{y_1} \int_{y_1}^1 \frac{dz_1}{z_1} \int_0^1 \frac{dy_2}{y_2} \int_{y_2}^1 \frac{dz_2}{z_2} \mathcal{K}^{(2)}(y_1, z_1, y_2, z_2) . \quad (16)$$

For the five different topologies the various products  $\Gamma_f^{(2)} \times \mathcal{K}^{(2)}$  of coupling factors and integration kernels are given by

$$\begin{aligned} \text{set (a):} & \quad \left[ \Gamma_{ff_1V_1}^2 \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \right] \left[ \Gamma_{f_1f_2V_2}^2 \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \right] \Theta(y_2 - y_1) , \\ \text{set (b):} & \quad -\Gamma_{ff_1V_1} \Gamma_{f_1f_2V_2} \Gamma_{f_2f_3V_1} \Gamma_{ff_3V_2} \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) , \\ \text{set (c):} & \quad \left[ \Gamma_{ff_1V_1}^2 \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \right] \left[ \Gamma_{ff_2V_2}^2 \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \right] , \\ \text{set (d):} & \quad \frac{1}{4} \Gamma_{ff_1V_1} \Gamma_{f_1f_2V_3} \Gamma_{ff_2V_2} G_{132} \left\{ \left[ \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) + \mathcal{K}^{(1)}(s, m_f^2, M_2, y_1, z_1) \right] \times \right. \\ & \quad \times \mathcal{K}^{(1)}(s, m_f^2, M_3, y_2, z_2) \Theta(y_2 - y_1) \left[ 1 + 3 \Theta(z_1 - z_2) \right] \\ & \quad + \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M_2, y_2, z_2) \times \\ & \quad \times \left[ 3 + \Theta(y_1 - y_2) \Theta(z_2 - z_1) + \Theta(y_2 - y_1) \Theta(z_1 - z_2) \right] \left. \right\} , \\ \text{set (e):} & \quad -\frac{1}{2} \Gamma_{ff_1V_1} \Gamma_{ff_1V_2} G_{134} G_{234} \left[ \mathcal{K}^{(1)}(s, m_f^2, M_3, y_2, z_2) + \mathcal{K}^{(1)}(s, m_f^2, M_4, y_2, z_2) \right] \times \\ & \quad \times \left[ \mathcal{K}^{(1)}(s, m_f^2, M_1, y_1, z_1) + \mathcal{K}^{(1)}(s, m_f^2, M_2, y_1, z_1) \right] \Theta(y_2 - y_1) \Theta(z_1 - z_2) . \quad (17) \end{aligned}$$

The totally antisymmetric coupling  $e G_{ijl}$  is the triple gauge-boson coupling with all three gauge-boson lines ( $i, j, l$ ) defined to be incoming at the interaction vertex. In our convention this coupling is fixed according to  $G_{\gamma W^+ W^-} = 1$  and  $G_{ZW^+ W^-} = -\cos \theta_w / \sin \theta_w$ . In the calculation of the integration kernel of set (e) we used the following couplings between the would-be Goldstone boson  $\phi$  and two gauge bosons:  $e C_{\phi W \gamma} = -e M_W$  and  $e C_{\phi W Z} = -e M_W \sin \theta_w / \cos \theta_w$ . The  $W^\pm - \phi^\pm$  mixing propagators, entering the same calculation, are given by

$$M_\mu^{\phi^\pm W^\pm}(k, n) = M_\mu^{W^\pm \phi^\pm}(k, n) = \frac{\mp i M_W}{k^2 - M_W^2 + i\epsilon} \frac{k^0 n_\mu}{\vec{k}^2} , \quad (18)$$

with  $k$  the momentum of the  $W^\pm$ -boson.

Note that several of the integration kernels involve a specific ordering in the energy variables  $z_i$  [in set (e) and part of set (d)] and/or the angular variables  $y_i$  [in sets (a),(e) and part of

set (d)]. The angular ordering in set (a), for instance, is caused by the fact that in the outer ( $k_1$ ) loop-integral the incoming fermion momentum is  $p$ , yielding  $(p - k_1)^2 \approx -s y_1$ , whereas in the inner ( $k_2$ ) loop-integral it is  $p - k_1$ , yielding  $(p - k_1 - k_2)^2 \approx -s(y_1 + y_2)$ . Therefore, only for  $y_2 \gg y_1$  a large logarithm develops. The energy ordering occurs, for instance, when the soft gauge-boson momentum of the outer loop-integral is the incoming momentum of the inner loop-integral [see set (e)]. The Sudakov parametrisation for  $k_2$  is based on  $k_1$  in that case and the large logarithms only develop if  $k_2^0 \ll k_1^0$ . The multitude of terms contributing to the integration kernels of sets (d) and (e) originate from the various soft gauge-boson regimes that are possible in those diagrams.

Note that certain diagrams look possible at first sight, but are in fact forbidden as a result of the charged current interactions of the  $W$  bosons. For instance, in set (b) it is not possible to exchange two  $W$  bosons without reversing the fermion-number flow (given by the direction of the Dirac propagator lines). Adding up all possible contributions, we find for the full two-loop Sudakov correction factor for right- and left-handed fermions/antifermions (see Sect. 2.3)

$$\delta_{f_R}^{(2)} = \delta_{\bar{f}_L}^{(2)} = \frac{1}{2} \left( \delta_{f_R}^{(1)} \right)^2 \quad (19a)$$

$$\delta_{f_L}^{(2)} = \delta_{\bar{f}_R}^{(2)} = \frac{1}{2} \left( \delta_{f_L}^{(1)} \right)^2. \quad (19b)$$

From Eqs. (19a) and (19b) we deduce our main statement, namely that the virtual electroweak two-loop Sudakov correction factor is obtained by a mere exponentiation of the one-loop Sudakov correction factor. This is in agreement with the corresponding assumptions in Ref. [7]. We also note that, in adding up all the contributions, we find that the ‘rainbow’ diagrams of set (a) yield the usual exponentiating terms plus an extra term for left-handed fermions similar to the one found in Ref. [6]. This extra term originates from the charged-current interactions and is only non-vanishing as a result of the mass gap between the massless photon and the massive  $Z$  boson. We therefore disagree with the statement in Ref. [7] that only rainbow diagrams contribute in physical gauges like the axial or the Coulomb gauge. Whereas in Ref. [6] the extra term was interpreted as a source of non-exponentiation, we observe that it in fact cancels against a specific term originating from the triple gauge-boson diagrams of set (d). Similar (gauge) cancellations take place between the ‘crossed rainbow’ diagrams of set (b), the reducible diagrams of set (c), and another part of the triple gauge-boson diagrams of set (d). Finally, the left-over terms of set (d) get cancelled by the contributions from the gauge-boson self-energy (‘frog’) diagrams of set (e). Hence, the cancellations that take place automatically in unbroken gauge theories also hold in the SM, in spite of it being a theory with more than one scale in the sense that the on-shell poles for photons and  $Z$  bosons do not coincide and therefore lead to different on-shell residues (see Sect. 2.3).

Comparing with the study in Ref. [5], we can make the following remark. A treatment of pure weak gauge-boson effects without reference to the photonic interactions breaks gauge-invariance, since the photon has an explicit  $SU(2)$  component. This holds even if the photon is treated fully inclusively as in Ref. [5]. Such a separation would require a very careful definition, for instance in terms of the typical energy regimes that govern the Sudakov effects of pure electromagnetic origin (ultrasoft energies:  $\lambda/\sqrt{s} \leq z \leq M/\sqrt{s}$ ) and collective electroweak origin (soft energies:  $M/\sqrt{s} \leq z \ll 1$ ).



## 2.3 Some useful integrals

In order to make our analysis more accessible, we present in this subsection the relevant loop-integrals, using the generic notation

$$I^{(i)} = \left(-\frac{\alpha}{\pi}\right)^i \int_0^1 \frac{dy_1}{y_1} \int_{y_1}^1 \frac{dz_1}{z_1} \dots \int_0^1 \frac{dy_i}{y_i} \int_{y_i}^1 \frac{dz_i}{z_i} \mathcal{K}^{(i)}(y_1, z_1, \dots, y_i, z_i) . \quad (20)$$

At one-loop level two types of kernels occur:

$$\begin{aligned} \mathcal{K}^{(1)}(s, m_f^2, M, y_1, z_1) &: I^{(1)} = L(M, M) , \\ \mathcal{K}^{(1)}(s, m_f^2, \lambda, y_1, z_1) &: I^{(1)} = L_\gamma(\lambda, m_f) . \end{aligned} \quad (21)$$

The functions  $L(M_1, M_2)$  and  $L_\gamma(\lambda, M_1)$  are the ones defined in Eqs. (13) and (14). At two-loop level seven new types of kernels occur. Four have angular ordering:

$$\begin{aligned} \mathcal{K}^{(1)}(s, m_f^2, M, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M, y_2, z_2) \Theta(y_2 - y_1) &: I^{(2)} = \frac{1}{2} L^2(M, M) , \\ \mathcal{K}^{(1)}(s, m_f^2, \lambda, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, \lambda, y_2, z_2) \Theta(y_2 - y_1) &: I^{(2)} = \frac{1}{2} L_\gamma^2(\lambda, m_f) , \\ \mathcal{K}^{(1)}(s, m_f^2, M, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, \lambda, y_2, z_2) \Theta(y_2 - y_1) &: I^{(2)} = \frac{7}{12} L^2(M, M) , \\ \mathcal{K}^{(1)}(s, m_f^2, \lambda, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M, y_2, z_2) \Theta(y_2 - y_1) &: I^{(2)} = L(M, M) L_\gamma(\lambda, m_f) \\ &\quad - \frac{7}{12} L^2(M, M) , \end{aligned} \quad (22)$$

and three have double ordering in both angle and energy:

$$\begin{aligned} \mathcal{K}^{(1)}(s, m_f^2, M, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M, y_2, z_2) \Theta(y_2 - y_1) \Theta(z_1 - z_2) &: I^{(2)} = \frac{1}{4} L^2(M, M) , \\ \mathcal{K}^{(1)}(s, m_f^2, M, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, \lambda, y_2, z_2) \Theta(y_2 - y_1) \Theta(z_1 - z_2) &: I^{(2)} = \frac{1}{3} L^2(M, M) , \\ \mathcal{K}^{(1)}(s, m_f^2, \lambda, y_1, z_1) \mathcal{K}^{(1)}(s, m_f^2, M, y_2, z_2) \Theta(y_2 - y_1) \Theta(z_1 - z_2) &: I^{(2)} = \frac{2}{3} L(M, M) L(M, m_f) \\ &\quad - \frac{1}{4} L^2(M, M) . \end{aligned} \quad (23)$$

Note that in the case of double ordering the collinear cut-off  $m_f^2$  of the  $y_2$  integral is in fact redundant.

## 3 Conclusions

In order to settle the controversy in the literature concerning the resummation of virtual Sudakov logarithms in the process  $e^+e^- \rightarrow f\bar{f}$ , we have calculated these Sudakov logarithms at one- and two-loop level in the Coulomb gauge. In this special gauge all the relevant contributions, involving the exchange of collinear-soft gauge bosons, are contained in the self-energies

of the external on-shell particles.

Our one-loop results are in agreement with the calculations in the literature, including the distinctive terms originating from the mass gap between the photon and the weak gauge bosons. At two-loop level our findings are consistent with an exponentiation of the one-loop results, in disagreement with claims in the literature that non-exponentiating terms should emerge at two-loop level. Based on these results and a similar study for transverse/longitudinal gauge-boson production processes, we conclude that as far as the balance between the one- and two-loop virtual Sudakov logarithms are concerned, the SM behaves like an unbroken theory at high energies. This conclusion can be extended to real-emission processes in a relatively straightforward way. After all, since the Sudakov logarithms originate from the exchange of soft, effectively on-shell gauge bosons, many of the features derived for the virtual corrections will be intimately related to properties of the corresponding real-emission processes.

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